1. (i) $2|(-1)^3| + 2|1^3| + 2|3^3| = 2 + 2 + 54 = 58.$

(ii) $\int_{-2}^{4} |x^3| dx = \int_{-2}^{0} (-x)^3 dx + \int_{0}^{4} x^3 dx = -\left[\frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{x^4}{4}\right]_{0}^{4} = 4 + 64 = 68.$

(iii) Riemann sum limit for $\int_0^1 x^7 dx = \left[\frac{x^8}{8}\right]_0^1 = \frac{1}{8}$

(iv) We have $\frac{x^4}{(x^2+5)} = (Ax+B)(x+1) + C + \frac{(Dx+E)(x+1)}{x^2+5}$. Letting $x \to -1$ gives $C = \frac{1}{6}$

2. (i) $\int_1^5 \Big(6y - y^2 - 5\Big) dy$

(ii) $2x\Big(\ln(1+\sqrt{1+x^2}) + \ln(1+\sqrt{1-x^2})\Big).$

(iii) $2\pi \int_0^1 x^2(1-x) dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}.$

3. (i) At $t = s$, $\frac{dy}{dx} = \frac{2-3s^2}{2s}$. So the tangent at $t = s$ is

$$y - 2s + s^3 = \frac{2-3s^2}{2s}(x - s^2).$$

Putting $x = 3$, $y = 0$ and cross multiplying gives $2s(-2s + s^3) = (2-3s^2)(3-s^2)$ which simplifies to $0 = s^4 - 7s^2 + 6 = (s^2-1)(s^2-6)$. The answer is $s = 1, -1, \sqrt{6}, -\sqrt{6}$.

(ii) $\frac{1}{2}\int_{-\pi}^{\pi} \Big(3 + 2\sin(\theta)\Big) d\theta = 3\pi.$

(iii) $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \sqrt{(e^\theta)^2 + (e^\theta)^2} d\theta = \sqrt{2} e^\theta d\theta$. The arclength is $\sqrt{2}\int_{-\pi}^{\pi} e^\theta d\theta = \sqrt{2}\Big(e^\pi - e^{-\pi}\Big).$

4. (i) This is a straight integration by parts

$$\int_0^1 x^2 \ln(x) dx = \left[\frac{x^3}{3}\ln(x)\right]_0^1 - \int_0^1 \frac{x^3}{3}\frac{1}{x} dx = 0 - \int_0^1 \frac{x^2}{3} dx = -\left[\frac{x^3}{9}\right]_0^1 = -\frac{1}{9}$$

(ii) The easiest way is

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)^2}{\cos(x)} dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos(x)^2}{\cos(x)} dx = \int_0^{\frac{\pi}{4}} \Big(\sec(x) - \cos(x)\Big) dx$$
$$= \left[\ln\Big(\sec(x) + \tan(x)\Big) - \sin(x)\right]_0^{\frac{\pi}{4}} = \ln\Big(\sqrt{2}+1\Big) - \frac{1}{\sqrt{2}}$$

Another effective way is to substitute $u = \sin(x)$ and then use partial fractions

$$\int_0^{\frac{\pi}{4}} \frac{\sin(x)^2}{\cos(x)} dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{u^2}{1-u^2} du = \int_0^{\frac{1}{\sqrt{2}}} \left(-1 - \frac{\frac{1}{2}}{1-u} + \frac{\frac{1}{2}}{1+u}\right) du$$
$$= \left[-u - \frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u)\right]_0^{\frac{1}{\sqrt{2}}}$$
$$= -\frac{1}{\sqrt{2}} - \frac{1}{2}\ln\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\ln\left(1 + \frac{1}{\sqrt{2}}\right)$$
$$= -\frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = -\frac{1}{\sqrt{2}} + \ln\Big(\sqrt{2}+1\Big)$$

(iii) We use the substitution $x+1=\sqrt{24}\tan(\theta)$ giving

$$\begin{aligned}
\int_0^4 \frac{x}{x^2+2x+25}dx &= \int_{\arctan\left(\frac{1}{\sqrt{24}}\right)}^{\arctan\left(\frac{5}{\sqrt{24}}\right)} \frac{\sqrt{24}\tan(\theta)-1}{\sqrt{24}\sec(\theta)}\sqrt{24}\sec(\theta)^2 d\theta \\
&= \int_{\arctan\left(\frac{1}{\sqrt{24}}\right)}^{\arctan\left(\frac{5}{\sqrt{24}}\right)} \left(\sqrt{24}\sec(\theta)\tan(\theta)-\sec(\theta)\right)d\theta \\
&= \left[\sqrt{24}\sec(\theta)-\ln(\sec(\theta)+\tan(\theta))\right]_{\arctan\left(\frac{1}{\sqrt{24}}\right)}^{\arctan\left(\frac{5}{\sqrt{24}}\right)} \\
&= \sqrt{24}\left(\sqrt{\frac{49}{24}}-\sqrt{\frac{25}{24}}\right)-\ln\left(\sqrt{\frac{25}{24}}+\sqrt{\frac{49}{24}}\right)+\ln\left(\sqrt{\frac{1}{24}}+\sqrt{\frac{25}{24}}\right) \\
&= 2-\ln(2)
\end{aligned}$$

but the hyperbolic substitution $x+1=\sqrt{24}\sinh(u)$ also works well

$$\begin{aligned}
\int_0^4 \frac{x}{x^2+2x+25}dx &= \int_{\operatorname{arcsinh}\left(\frac{1}{\sqrt{24}}\right)}^{\operatorname{arcsinh}\left(\frac{5}{\sqrt{24}}\right)} \frac{\sqrt{24}\sinh(u)-1}{\sqrt{24}\cosh(u)}\sqrt{24}\cosh(u)du \\
&= \int_{\operatorname{arcsinh}\left(\frac{1}{\sqrt{24}}\right)}^{\operatorname{arcsinh}\left(\frac{5}{\sqrt{24}}\right)} \left(\sqrt{24}\sinh(u)-1\right)du \\
&= \left[\sqrt{24}\cosh(u)-u\right]_{\operatorname{arcsinh}\left(\frac{1}{\sqrt{24}}\right)}^{\operatorname{arcsinh}\left(\frac{5}{\sqrt{24}}\right)} \\
&= \sqrt{24}\left(\sqrt{\frac{49}{24}}-\sqrt{\frac{25}{24}}\right)-\operatorname{arcsinh}\left(\frac{5}{\sqrt{24}}\right)+\operatorname{arcsinh}\left(\frac{1}{\sqrt{24}}\right) \\
&= (7-5)-\ln\left(\sqrt{\frac{25}{24}}+\sqrt{\frac{49}{24}}\right)+\ln\left(\sqrt{\frac{1}{24}}+\sqrt{\frac{25}{24}}\right) \\
&= 2-\ln(2)
\end{aligned}$$

5. We get from $x=3t^{\frac{1}{2}}-t^{\frac{3}{2}}$, $t=3t$,

$$\frac{dx}{dt}=\frac{3}{2}\left(t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right),\quad \frac{dy}{dt}=3$$

which leads to

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{3}{2}\right)^2(t^{-1}-2+t+4)=\left(\frac{3}{2}\right)^2(t^{-1}+2+t)=\left(\frac{3}{2}\left(t^{-\frac{1}{2}}+t^{\frac{1}{2}}\right)\right)^2$$

and $ds=\frac{3}{2}\left(t^{-\frac{1}{2}}+t^{\frac{1}{2}}\right)dt$ for $t>0$. The desired surface area is then

$$\begin{aligned}
2\pi\int_{t=1}^2 xds &= 3\pi\int_1^2\left(3t^{\frac{1}{2}}-t^{\frac{3}{2}}\right)\left(t^{-\frac{1}{2}}+t^{\frac{1}{2}}\right)dt = 3\pi\int_1^2\left(3+2t-t^2\right)dt \\
&= 3\pi\left[3t+t^2-\frac{1}{3}t^3\right]_1^2 = 11\pi
\end{aligned}$$

6. The required volume is

$$\pi\int_{x=0}^{\pi} y^2dx=\pi\int_{x=0}^{\pi} x^2(1+\cos(x))^2dx.$$

The integral is computed by splitting and then integrating by parts twice.

$$
\begin{aligned}
\pi \int_0^\pi x^2(1+\cos(x))^2 dx &= \pi \int_0^\pi x^2(1+2\cos(x)+\cos(x)^2)dx \\
&= \frac{\pi}{2}\int_0^\pi x^2(2+4\cos(x)+1+\cos(2x))dx \\
&= \frac{\pi}{2}\int_0^\pi x^2(3+4\cos(x)+\cos(2x))dx \\
&= \frac{\pi}{2}\left[x^3\right]_0^\pi + \frac{\pi}{2}\left[x^2\Big(4\sin(x)+\frac{1}{2}\sin(2x)\Big)\right]_0^\pi - \frac{\pi}{2}\int_0^\pi 2x\Big(4\sin(x)+\frac{1}{2}\sin(2x)\Big)dx \\
&= \frac{\pi^4}{2} - \pi\int_0^\pi x\Big(4\sin(x)+\frac{1}{2}\sin(2x)\Big)dx \\
&= \frac{\pi^4}{2} + \pi\left[x\Big(4\cos(x)+\frac{1}{4}\cos(2x)\Big)\right]_0^\pi - \pi\int_0^\pi \Big(4\cos(x)+\frac{1}{4}\cos(2x)\Big)dx \\
&= \frac{\pi^4}{2} + \pi^2\left(-4+\frac{1}{4}\right) - \pi\Big[4\sin(x)+\frac{1}{8}\sin(2x)\Big]_0^\pi \\
&= \frac{1}{2}\pi^4 - \frac{15}{4}\pi^2
\end{aligned}
$$

7. (i) $\displaystyle\sum_{n=1}^{\infty} n\arctan\left(\frac{1}{n^2}\right)$ diverges by limit comparison with the harmonic series. We have

$$\lim_{n\to\infty}\frac{n\arctan\left(\frac{1}{n^2}\right)}{\frac{1}{n}} = \lim_{x\to 0+}\frac{\arctan(x)}{x}$$

if the limit on the right exists. This is obtained from putting $x = n^{-2}$. But the limit on the right is of type $0/0$ and L'Hopital's rule gives

$$\lim_{x\to 0+}\frac{\arctan(x)}{x} = \lim_{x\to 0+}\frac{1}{1+x^2} = 1.$$

(ii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(n+2)^n}{n^{n+2}}$ converges absolutely in limit comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ since

$$\frac{(n+2)^n}{n^{n+2}} = \left(1+\frac{2}{n}\right)^n\frac{1}{n^2}$$

and since

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$$